

Tomographic entropic inequalities in the probability representation of quantum mechanics

Margarita A. Man'ko and Vladimir I. Man'ko

P. N. Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia

Abstract. A review of the tomographic-probability representation of classical and quantum states is presented. The tomographic entropies and entropic uncertainty relations are discussed in connection with ambiguities in the interpretation of the state tomograms which are considered either as a set of the probability distributions of random variables depending on extra parameters or as a single joint probability distribution of these random variables and random parameters with specific properties of the marginals. Examples of optical tomograms of photon states, symplectic tomograms, and unitary spin tomograms of qudits are given. A new universal integral inequality for generic wave function is obtained on the base of tomographic entropic uncertainty relations.

Keywords: Entropy, uncertainty relations, modified tomogram, universal inequality for the wave function.

PACS: 42.50.Ct, 42.50.Ex, 02.50.Cw

INTRODUCTION

In a conventional approach, the states of quantum systems are described by the wave function [1], the density matrix [2, 3] or a vector in the Hilbert space [4]. In the probability representation of quantum mechanics [5], the states of quantum systems are described by the tomographic-probability distributions or tomograms (see, for example, the review [6]). It was shown [7, 8] that the states of classical systems can also be described by the tomograms used as alternatives to the probability densities on the phase space.

Since any probability distribution provides such informational and statistical characteristics as Shannon [9] and R nyi [10] entropies, the corresponding tomographic entropies were introduced [7, 11], and the properties of such entropies were discussed (see, for example, [12]). Among these properties, there are entropic uncertainty relations known for the probability distributions associated with the wave functions (see, for example, [13, 14]) and density matrices (see, for example, [15]). Recently, the tomographic entropic uncertainty relations obtained in [16, 12] have been confirmed experimentally [17] with an accuracy of a few percents.

The tomograms for both the continuous photon quadratures [5] and the discrete spin variables [18, 19] have some specific properties. The tomograms are the distribution functions of random variables, and these functions depend on extra parameters. For photon optical tomograms [20, 21], the probability distribution $w(X, \theta)$ depends on a random homodyne real quadrature $-\infty < X < \infty$ and the local oscillator phase $0 \leq \theta \leq 2\pi$ considered as a control parameter. For unitary spin tomogram $w(m, u)$ [22], the function depends on a random discrete variable $-j \leq m \leq j$, which is the spin projection on the quantization axes obtained by the action of a unitary rotation matrix u on the initial z axis.

The aim of this paper is to demonstrate some ambiguities in such interpretation. We present the other possibility to interpret the tomograms as joint probability distributions of two random variables — considering the extra parameters as random variables. For optical tomograms, such interpretation was discussed in [23, 17]. Such interpretation gives some new clarification of the properties of classical and quantum tomograms and provides with possibilities of the generalization of known tomograms by introducing many other joint probability distributions containing the same information on quantum (also classical) states as the initial optical or symplectic tomograms for the continuous homodyne quadratures and discrete spin variables for spin tomograms. The other goal is to reformulate some entropic uncertainty relations for pure states as a new integral inequality for the wave functions.

ENTROPIES IN THE PROBABILITY THEORY

In the information-theory context, entropy is related to an arbitrary probability-distribution function. We remind the notion of Shannon entropy [9]. Given the probability distribution $P(n)$, where n is a discrete random variable, i.e.,

$P(n) \geq 0$, and the normalization condition holds $\sum_n P(n) = 1$, one has, by definition, the Shannon entropy

$$S = -\sum_n P(n) \ln P(n) = -\langle \ln P(n) \rangle. \quad (1)$$

There exist other kinds of entropies depending on extra parameter q and associated to the probability distribution $P(n)$, for example, Rényi entropy [10]

$$R(q) = \frac{1}{1-q} \ln \left[\sum_n \left(P(n) \right)^q \right], \quad q > 0. \quad (2)$$

In the limit $q \rightarrow 1$, one has the equality of the Rényi entropy to the Shannon entropy $R(1) = S$. The Shannon entropy S is a number. The Rényi entropy $R(q)$ is the function of a parameter q and, due to this, it contains more information on details of the probability distribution $P(n)$ including the value of Shannon entropy.

For distribution functions $P(x)$, in the case of continuous random variable x , one has the same definition with the replacements $n \rightarrow x$ and $\sum_n \rightarrow \int dx$. For the case of quantum-system states with density matrix ρ , one has an analog of the Shannon entropy, which is called the von Neumann entropy, given by formula with taking the trace

$$S_{\text{vN}} = -\text{Tr}(\rho \ln \rho), \quad (3)$$

and quantum Rényi entropy

$$R_\rho(q) = \frac{1}{1-q} \ln \text{Tr}(\rho^q). \quad (4)$$

The above formulas generalize the definition of entropies in the classical domain. In the limit $q \rightarrow 1$, for quantum system entropies, one has an equality analogous to the previous equality $R(1) = S$, i.e., one has the equality of the quantum Rényi entropy to the von Neumann entropy $R_\rho(1) = S_{\text{vN}}$.

Below we apply the introduced definitions of entropy in all cases of probability distributions and density matrices to tomograms, since the tomograms themselves are fair probability distributions for both classical and quantum systems.

SPECIFIC PROBABILITY DISTRIBUTIONS AND THEIR MARGINALS

Now we consider some special probability distributions $P(a, b)$ of a random variable a , which depends also on extra parameters b , where a and b denote some sets of variables, both continuous and discrete ones. The functions $P(a, b)$ are nonnegative $P(a, b) \geq 0$ and normalized $\sum_a P(a, b) = 1$ for arbitrary values of the parameters b . This property of the function $P(a, b)$ is called “no signaling.” The functions $P(a, b)$ can be obtained from a joint probability distribution $f(a, b) \geq 0$ of two random variables satisfying the normalization condition $\sum_{a,b} f(a, b) = 1$. In fact, one can construct the marginal $K(b) = \sum_a f(a, b)$ and define the function $P(a, b)$ by the relation $P(a, b) = f(a, b)K^{-1}(b)$. In this case, the function $P(a, b)$ is called the conditional probability distribution of random variable a provided the output of the second event is known.

A simple example of such a function is the case where $a = 1, 2$, $b = 1, 2$, $P(1, 1) = x$, $P(1, 2) = 1 - x$, $P(2, 1) = y$, $P(2, 2) = 1 - y$, and $0 \leq x, y \leq 1$. In this example, the function $P(a, b)$ corresponds to the set of two different probability distributions $P(a, 1)$, such that $P(1, 1) = x$ and $P(2, 1) = 1 - x$, and $P(a, 2)$, such that $P(1, 2) = y$ and $P(2, 2) = 1 - y$. The structure of this function provides the possibility to consider it as a single joint probability distribution $\mathcal{P}(a, b)$ of two random variables determined by the formula

$$\mathcal{P}(a, b) = P(a, b)/2. \quad (5)$$

In fact, $\mathcal{P}(a, b) \geq 0$ and $\sum_{a,b} \mathcal{P}(a, b) = 1$. This joint probability distribution has two marginals, namely,

$$\Pi_1(a) = \sum_b \mathcal{P}(a, b), \quad \Pi_2(b) = \sum_a \mathcal{P}(a, b). \quad (6)$$

One can see that

$$\Pi_1(1) = (x + y)/2, \quad \Pi_1(2) = 1 - (x/2) - (y/2), \quad (7)$$

$$\Pi_2(1) = 1/2, \quad \Pi_2(2) = 1/2. \quad (8)$$

If one considers the function $\Pi_2(b)$ as a probability distribution, it corresponds to maximum chaotic behavior of the random variable b with maximum Shannon entropy $S = \ln 2$.

The generic joint probability distribution is determined by three nonnegative parameters p_1 , p_2 , and p_3 as follows:

$$\Pi(1,1) = p_1, \quad \Pi(1,2) = p_2, \quad \Pi(2,1) = p_3, \quad \Pi(2,2) = 1 - p_1 - p_2 - p_3. \quad (9)$$

The above distributions, for which one of the marginals coincides with the distribution with maximum entropy, is determined by two parameters. Thus, we can see that the joint probability distributions with such specific properties of the marginals belong to a subdomain in the simplex corresponding to a set of generic probability distributions of two random variables. It is clear that an analogous situation can be found for the other functions and the other domains of the variables a and b . Also, for the function $P(a,b)$, a new joint probability distribution $W(a,b)$ can be constructed using an arbitrary probability distribution $w(b)$ of a random variable b as follows: $W(a,b) = P(a,b)w(b)$.

In fact, $W(a,b) \geq 0$ and $\sum_{a,b} W(a,b) = 1$, since $\sum_b w(b) = 1$, as well as $\sum_a P(a,b) = 1$. The permutation symmetry $a \rightleftharpoons b$ can take place.

The other examples of optical, symplectic and spin tomograms, which have analogous “no signaling” properties, are given in the next sections.

OPTICAL AND SYMPLECTIC TOMOGRAMS

Tomographic-probability distributions of classical particles

Given the probability density $f(q,p)$ on the phase space. The function $f(q,p)$, due to the physical meaning of the probability distribution, is nonnegative and normalized $\int f(q,p) dq dp = 1$. Let us calculate the marginal probability density of the particle's position X in a rotated reference frame on the phase space with new rotated axes. One has the expression for position X in rotated reference frame as follows:

$$X = q \cos \theta + p \sin \theta, \quad (10)$$

where θ is the rotation angle. One can see that for $\theta = 0$, $X = q$ and for $\theta = \pi/2$, $X = p$.

The marginal probability density $w(X, \theta)$ (called optical tomogram in quantum optics, but we will use this name also in classical statistical mechanics) reads

$$w(X, \theta) = \langle \delta(X - q \cos \theta - p \sin \theta) \rangle = \int f(q,p) \delta(X - q \cos \theta - p \sin \theta) dq dp. \quad (11)$$

Now we introduce another tomogram (related to the optical tomogram) accompanying the rotation of reference frame in the phase space by scaling the position and momentum before the rotation. Namely, we consider the marginal probability density denoted as $M(X, \mu, \nu)$ of the particle's position X in a reference frame on the phase space, which first was rescaled and then was rotated; it reads

$$M(X, \mu, \nu) = \langle \delta(X - \mu q - \nu p) \rangle = \int f(q,p) \delta(X - \mu q - \nu p) dq dp, \quad (12)$$

where μ and ν could be arbitrary real numbers. The probability distribution $M(X, \mu, \nu)$ is called symplectic tomogram of the classical particle's state. It is normalized $\int M(X, \mu, \nu) dX = 1$, due to the property of delta-function $\int \delta(X - \mu q - \nu p) dX = 1$ and the normalization of the distribution $f(q,p)$ on the phase space. Due to the homogeneity of the Dirac delta-function, i.e., $\delta(\lambda x) = |\lambda|^{-1} \delta(x)$, the symplectic tomogram is also the homogeneous function, $M(\lambda X, \lambda \mu, \lambda \nu) = |\lambda|^{-1} M(X, \mu, \nu)$. Thus, one has the connection between the optical and symplectic tomograms, due to the homogeneity property, i.e.,

$$w(X, \theta) = M(X, \cos \theta, \sin \theta), \quad (13)$$

$$M(X, \mu, \nu) = \frac{1}{\sqrt{\mu^2 + \nu^2}} w\left(\frac{X}{\sqrt{\mu^2 + \nu^2}}, \tan^{-1} \frac{\nu}{\mu}\right). \quad (14)$$

Formulae for the optical tomogram, given by its definition, namely, (11), and symplectic tomogram, given by its definition (12), turn out to coincide with the well-known integral Radon transform of the function of two variables $f(q, p)$, which has the inverse. The inverse reads

$$f(q, p) = \frac{1}{4\pi^2} \int M(X, \mu, \nu) e^{i(X - \mu q - \nu p)} dX d\mu d\nu \geq 0. \quad (15)$$

The probability distribution $M(X, \mu, \nu)$ can be used to calculate momenta of the random variables q and p . In fact, due to the physical meaning of the marginal probability distribution $M(X, \mu, \nu)$, one has

$$\langle q^n \rangle = \int M(X, 1, 0) X^n dX, \quad \langle p^n \rangle = \int M(X, 0, 1) X^n dX. \quad (16)$$

Tomographic-probability distributions of quantum particles

The quantum particle's state can be described by the tomogram obtained using the formula for classical tomogram with averaging the delta-function (11). So, we start from this formula keeping only the form with averaging $w(X, \theta) = \langle \delta(X - q \cos \theta - p \sin \theta) \rangle$, but with the following replacement in this form the numbers q and p by the corresponding operators, i.e., the position is replaced by the position operator $q \rightarrow \hat{q}$, and the momentum is replaced by the momentum operator $p \rightarrow \hat{p}$. Also the classical averaging has to be replaced with averaging by means of the quantum-state density operator $\hat{\rho}$.

For photon states, the photon quadrature components play the role of position q and momentum p . Then, for the photon quantum state in quantum optics, the optical tomogram is defined as

$$w(X, \theta) = \langle \delta(X - \hat{q} \cos \theta - \hat{p} \sin \theta) \rangle. \quad (17)$$

The average means that we replaced the probability distribution $f(q, p)$ in the definition of classical optical tomogram by the density operator, i.e., $f(q, p) \rightarrow \hat{\rho}$ and applied the formula for average of the operator \hat{A} of the form $\langle \hat{A} \rangle = \text{Tr}(\hat{\rho} \hat{A})$.

The definition of optical tomogram (17) can be done in a more known form (see, [20, 21]) which uses the Wigner function $W(q, p)$ of the photon quantum state

$$w(X, \theta) = \int W(q, p) \delta(X - q \cos \theta - p \sin \theta) \frac{dq dp}{2\pi}.$$

One can see that the optical tomogram of classical particle is given by the same formula with replacement $W(q, p)/2\pi \rightarrow f(q, p)$. The above optical tomogram of the photon quantum state is measured by homodyne detector [24].

The symplectic tomogram of the quantum state is given by the classical formula (12) with the same replacements $q \rightarrow \hat{q}$ and $p \rightarrow \hat{p}$, i.e.,

$$M(X, \mu, \nu) = \langle \delta(X - \mu \hat{q} - \nu \hat{p}) \rangle. \quad (18)$$

The quantum tomogram $M(X, \mu, \nu)$ determines the density operator $\hat{\rho}$ by the formula analogous to the classical formula for reconstructing the probability distribution $f(q, p)$ on the phase space but with the replacement $f(q, p) \rightarrow \hat{\rho}$, $q \rightarrow \hat{q}$, $p \rightarrow \hat{p}$, and $1/4\pi^2 \rightarrow 1/2\pi$, i.e.,

$$\hat{\rho} = \frac{1}{2\pi} \int M(X, \mu, \nu) e^{i(X - \mu \hat{q} - \nu \hat{p})} dX d\mu d\nu. \quad (19)$$

One can see that inverse Radon transform (15) for the classical symplectic tomogram $M(X, \mu, \nu)$ coincides with its Fourier transform. Reconstruction formula (19) for the quantum density operator $\hat{\rho}$ has the form of "quantized" Fourier transform of the quantum symplectic tomogram $M(X, \mu, \nu)$.

We summarize the notion of classical and quantum states in terms of tomograms $M(X, \mu, \nu)$ in the tomographic-probability representation as follows.

- The states in both classical and quantum mechanics can be associated with nonnegative normalized homogeneous probability distributions $M(X, \mu, \nu)$ (tomograms) depending on a random variable X and real parameters μ and ν .

- The quantum optical and symplectic tomograms satisfy the same formulae (13) and (14) like the classical tomograms. This means that measuring the quantum optical tomogram $w(X, \theta)$ by homodyne detector implies measuring the symplectic tomogram. Namely in homodyne experiments one can study optical tomograms and entropic inequalities which distinguish the classical and quantum domains.

MODIFIED OPTICAL, SYMPLECTIC, AND SPIN TOMOGRAMS

The optical and symplectic tomograms introduced have a form of the function $P(a, b)$ discussed in the previous sections. In fact, for the optical tomogram the variable a is the homodyne quadrature X , and the variable b is the local oscillator phase θ .

So we can introduce a modified optical tomogram

$$W(X, \theta) = w(X, \theta)R(\theta), \quad (20)$$

where $R(\theta) \geq 0$ and $\int_0^{2\pi} R(\theta) d\theta = 1$. Thus, $R(\theta)$ is an arbitrary probability density on a circle; for example, we can use $R(\theta) = (2\pi)^{-1}$.

For symplectic tomogram, one can provide a modification of the form

$$\tilde{M}(X, \mu, \nu) = M(X, \mu, \nu)R(\mu, \nu), \quad (21)$$

where $R(\mu, \nu) \geq 0$ and $\iint R(\mu, \nu) d\mu d\nu = 1$. Thus, $R(\mu, \nu)$ can be taken as an arbitrary probability density on the plane (μ, ν) . For example, we can use the Gaussian distribution function.

Summarizing, for both the classical and quantum cases, we have introduced a modified optical tomogram which is the joint probability distribution of the homodyne quadrature component and the local oscillator phase. For symplectic tomogram of the classical state, one can introduce a modified version of the Gaussian form

$$\tilde{M}_G(X, \mu, \nu) = \frac{1}{\pi} \int f(q, p) [\delta(X - \mu q - \nu p) \exp(-\mu^2 - \nu^2)] dq dp. \quad (22)$$

The inversion formula reads

$$f(q, p) = \frac{1}{4\pi} \int \tilde{M}_G(X, \mu, \nu) \exp[\mu^2 + \nu^2 + i(X - \mu q - \nu p)] dX d\mu d\nu. \quad (23)$$

For the quantum case, the modified optical tomogram reads

$$W(X, \theta) = \langle \delta(X - \hat{q} \cos \theta - \hat{p} \sin \theta) R(\theta) \rangle. \quad (24)$$

The modified symplectic tomogram of quantum state can be defined using the Gaussian factor as follows:

$$\tilde{M}(X, \mu, \nu) = \frac{1}{\pi} \langle \delta(X - \mu \hat{q} - \nu \hat{p}) \exp(-\mu^2 - \nu^2) \rangle, \quad (25)$$

and the inverse of (25) is

$$\hat{\rho} = \frac{1}{2} \int \tilde{M}(X, \mu, \nu) \exp[\mu^2 + \nu^2 + i(X - \mu \hat{q} - \nu \hat{p})] dX d\mu d\nu. \quad (26)$$

One can also make a modification of the same kind of the unitary spin tomogram. The tomogram $w(m, u)$ reads [18, 19, 25, 26]

$$w(m, u) = \langle m | u \hat{\rho} u^\dagger | m \rangle, \quad (27)$$

and it is the function of spin projection $-j \leq m \leq j$ and the unitary-group element u .

If the matrix u coincides with the matrix of irreducible representation of the group $SU(2)$, the tomogram is the function $w(m, \vec{n})$ of the spin projection m depending on the quantization direction \vec{n} . The spin tomograms $w(m, u)$ and $w(m, \vec{n})$ are nonnegative and normalized functions

$$\sum_{m=-j}^j w(m, u) = 1, \quad \sum_{m=-j}^j w(m, \vec{n}) = 1 \quad (28)$$

for arbitrary directions \vec{n} and arbitrary unitary matrices u . This means that the tomograms belong to the set of functions $P(a, b)$, which can be related to functions $\mathcal{P}(a, b)$ discussed above. In view of this, one can introduce modified spin tomograms. One of the modifications reads

$$\tilde{w}(m, \vec{n}) = w(m, \vec{n})R(\vec{n}), \quad (29)$$

where $R(\vec{n})$ is any probability density on the sphere S^2 , i.e., $R(\vec{n}) \geq 0$ and the integral over the sphere $\int R(\vec{n}) d\vec{n} = 1$.

The modified unitary spin tomogram reads

$$\tilde{w}(m, u) = w(m, u)R(u), \quad (30)$$

where $R(u)$ is any probability density on the unitary group, i.e., $R(u) \geq 0$ and $\int R(u) du = 1$, with du being the Haar measure on the group, $\int du = V$, and V the volume on the unitary group. For example, one can consider a maximum chaotic distribution $R(u) = 1/V$ with the Shannon entropy $S_u = \ln V$.

In the case of modified spin tomogram $\tilde{w}(m, \vec{n})$, we can take the distribution $R(\vec{n}) = 1/4\pi$ corresponding to the area of the unit-radius sphere $\int d\vec{n} = 4\pi$. This maximum chaotic distribution has the Shannon entropy $S_{\vec{n}} = \ln 4\pi$.

Thus, we introduced the modified spin tomograms, which are functions of two sets of random variable corresponding to functions $P(a, b)$, where a is the spin projection m , and b is either a point on the unit sphere S^2 parametrized by the unit vector \vec{n} or the element of the unitary group. It is worth noting that all other available tomographic-probability distributions like the photon-number tomograms [27] or the center-of-mass tomograms [28] can also be modified in an analogous way. One can see that there exists an ambiguity in choosing the tomographic-probability distributions which can be associated with the states in both classical and quantum domains. The ambiguity is related to the choice of the probability distribution of random parameters.

MODIFIED TOMOGRAPHIC ENTROPIES

Since the symplectic tomogram is the standard probability distribution, one can introduce entropy associated with the tomogram of quantum state [7] or with the tomogram of analytic signal [11]. Thus one has entropy as the function of two real variables μ and ν

$$S(\mu, \nu) = - \int M(X, \mu, \nu) \ln M(X, \mu, \nu) dX. \quad (31)$$

We call this entropy the symplectic entropy. In view of the homogeneity and normalization conditions for tomograms, one has the additivity property

$$S(\lambda\mu, \lambda\nu) = S(\mu, \nu) + \ln |\lambda|. \quad (32)$$

Also one has the optical tomographic entropy associated with the optical tomogram $w(X, \theta)$ as

$$S(\theta) = - \int w(X, \theta) \ln w(X, \theta) dX, \quad (33)$$

and this entropy depends on local oscillator phase in experiments with measuring photon homodyne quadrature.

Since we introduced the modified optical and symplectic tomograms, modified tomographic entropies can be defined.

For symplectic tomogram, modified tomographic entropy reads

$$S^{(\text{sym})} = \langle S(\mu, \nu) \rangle + S_R^{(\text{sym})}, \quad (34)$$

where

$$\langle S(\mu, \nu) \rangle = \int d\mu d\nu R(\mu, \nu) S(\mu, \nu), \quad (35)$$

$$S_R = - \int R(\mu, \nu) \ln R(\mu, \nu) d\mu d\nu. \quad (36)$$

For optical tomogram, modified tomographic entropy is

$$S^{(\text{opt})} = \langle S(\theta) \rangle + S_R^{(\text{opt})}, \quad (37)$$

where

$$\langle S(\theta) \rangle = \int_0^{2\pi} d\theta S(\theta) R(\theta), \quad (38)$$

$$S_R^{(\text{opt})} = - \int_0^{2\pi} R(\theta) \ln(R(\theta)) d\theta. \quad (39)$$

Analogous modified tomographic entropy can be defined for spin tomograms.

The quantum optical tomogram of the pure state is determined by the wave function as (see, for example, [12])

$$w(X, \theta) = \left| \int \psi(y) \exp \left[\frac{i}{2} \left(\cot \theta (y^2 + X^2) - \frac{2X}{\sin \theta} y \right) \right] \frac{dy}{\sqrt{2\pi i \sin \theta}} \right|^2. \quad (40)$$

On the other hand, this tomogram formally equals to

$$w(X, \theta) = |\psi(X, \theta)|^2, \quad (41)$$

where the wave function reads

$$\psi(X, \theta) = \frac{1}{\sqrt{2\pi i \sin \theta}} \int \exp \left[\frac{i}{2} \left(\cot \theta (y^2 + X^2) - \frac{2X}{\sin \theta} y \right) \right] \psi(y) dy, \quad (42)$$

being the fractional Fourier transform of the wave function $\psi(y)$. This wave function corresponds to the wave function of a harmonic oscillator with $\hbar = m = \omega = 1$ taken at the “time” moment θ provided the wave function at the initial time moment $\theta = 0$ equals to $\psi(y)$.

In view of expressions of tomogram in terms of the wave function (41) and (42), one has the entropic uncertainty relation in the form

$$S(\theta) + S(\theta + \pi/2) \geq \ln \pi e, \quad (43)$$

which is the Hirshman uncertainty relation

$$- \int |\psi(x)|^2 \ln |\psi(x)|^2 dx - \int |\tilde{\psi}(p)|^2 \ln |\tilde{\psi}(p)|^2 dp \geq \ln \pi e, \quad (44)$$

considered in a rotated reference frame on the phase space [13, 12], with $\tilde{\psi}(p)$ being the wave function in the momentum representation. In (43), $S(\theta)$ is the tomographic Shannon entropy associated with optical tomogram (40) which is measured by homodyne detector.

One can write the subadditivity and strong subadditivity conditions for modified spin tomograms. For example, using Eq. (29), we obtain the subadditivity condition of the form

$$- \sum_{\vec{n}} \left(\sum_m \tilde{w}(m, \vec{n}) \ln \left[\sum_m \tilde{w}(m, \vec{n}) \right] \right) - \sum_m \left(\sum_{\vec{n}} \tilde{w}(m, \vec{n}) \ln \left[\sum_{\vec{n}} \tilde{w}(m, \vec{n}) \right] \right) \geq - \sum_m \sum_{\vec{n}} \tilde{w}(m, \vec{n}) \ln \tilde{w}(m, \vec{n}), \quad (45)$$

where we used several (arbitrary number) different directions \vec{n} such that $\sum_{\vec{n}} R(\vec{n}) = 1$.

For two qudits, the modified tomogram of the state with density matrix $\rho(1, 2)$ can be given as

$$\tilde{w}(m_1, m_2, u) = \langle m_1 m_2 | u \rho(1, 2) u^\dagger | m_1 m_2 \rangle R(u), \quad (46)$$

where for the distribution $R(u)$ one can take several (arbitrary number) different matrices u such that $\sum_u R(u) = 1$. One has the strong subadditivity condition

$$S(1, 2) + S(2, 3) \geq S(1, 2, 3) + S(2), \quad (47)$$

where $S(1,2)$ and $S(2,3)$ are Shannon entropies for marginal distributions

$$\tilde{\Omega}(m_1, u) = \sum_{m_2} \tilde{w}(m_1, m_2, u) \quad \text{and} \quad \tilde{\Omega}(m_2, u) = \sum_{m_1} \tilde{w}(m_1, m_2, u), \quad (48)$$

respectively. The entropy $S(1,2,3)$ is the Shannon entropy for distribution (46) and $S(2)$ is the Shannon entropy for distribution $\tilde{\Omega}(u) = \sum_{m_1, m_2} \tilde{w}(m_1, m_2, u)$.

Using (40) and integrating (43) over the local oscillator phase $0 \leq \theta \leq 2\pi$, we obtain the inequality

$$\begin{aligned} & - \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{d\theta dX}{|\sin \theta|} \left| \int_{-\infty}^{\infty} \psi(y) \exp \left(\frac{i \cot \theta}{2} y^2 - \frac{iXy}{\sin \theta} \right) dy \right|^2 \\ & \times \ln \left[\frac{1}{2\pi |\sin \theta|} \left| \int_{-\infty}^{\infty} \psi(z) \exp \left(\frac{i \cot \theta}{2} z^2 - \frac{iXz}{\sin \theta} \right) dz \right|^2 \right] \geq 2\pi^2 \ln \pi e. \end{aligned} \quad (49)$$

This universal integral inequality must be fulfilled for an arbitrary wave function $\psi(y)$, satisfying the normalization condition $\int_{-\infty}^{\infty} |\psi(y)|^2 dy = 1$. The entropic inequality in the form of inequality for the zero Fourier component of the function of θ in Eq. (43) was obtained in [29] and in the form of integral inequality containing the optical tomogram and checked experimentally in [17].

Entropic inequality (49) is obvious in the tomographic-probability representation of quantum states but in the standard formulation of quantum mechanics it becomes more complicated integral inequality for the wave function. This inequality could be related either to the properties of optical tomograms considered as the function of one random variable X or as the joint probability distribution of random variables X and θ .

CONCLUSIONS

To conclude, we summarize the main results of our work.

- We showed that all the available state tomograms can be considered either as the probability distributions of random variables depending on extra parameters with no signaling properties or as the joint probability distributions of both sets of variables and the parameters.
- We presented possible modifications of optical, symplectic, and spin tomograms.
- We studied properties of the wave function $\psi(y) \in L_2$ for the available optical tomographic entropic inequality associated with the tomograms and obtained the universal integral inequality for an arbitrary wave function.
- We clarified the ambiguity in choosing the tomographic-probability distribution describing the states in both the classical and quantum domains.

ACKNOWLEDGMENTS

This study was supported by the Russian Foundation for Basic Research under Projects Nos. 10-02-00312 and 11-02-00456. The authors thank the Universidad Nacional Autonoma de Mexico and Prof. Octavio Castaños and the Organizers of the conference “Beauty in Physics: Theory and Experiment” (the Hacienda Cocoyoc, Morelos, Mexico, May 14–18, 2012) Profs. Alejandro Frank and Roelof Bijker for invitation and kind hospitality.

REFERENCES

1. E. Schrödinger, “Der stetige Übergang von der Mikro- zur Makromechanik,” *Naturwissenschaften*, **14**, 664–666 (1926) [“The continuous transition from micro- to macromechanics,” in E. Schrödinger, *Collected papers on Wave Mechanics*, the second edition, Chelsea Publ. Corp., New York, 1978, pp. 41–44].
2. L. Landau, “Das Dämpfungsproblem in der Wellenmechanik,” *Z. Phys.*, **45**, 430–441 (1927) [“The damping problem in wave mechanics,” in *Collected Papers of L. D. Landau*, edited by D. Ter Haar, Gordon & Breach, New York, 1965, pp. 8–18].
3. J. von Neumann, “Wahrscheinlichkeitstheoretischer Aufbau der Quantenmechanik,” *Göttingen Nachrichten*, **11**, S. 245–272 (November 1927).

4. P. A. M. Dirac, *Principles of Quantum Mechanics*, 4th ed., Oxford University Press, London, UK (1958).
5. S. Mancini, V. I. Man'ko, and P. Tombesi, "Symplectic tomography as classical approach to quantum systems," *Phys. Lett. A* **213**, 1–6 (1996).
6. A. Ibort, V. I. Man'ko, G. Marmo, A. Simoni, and F. Ventriglia, "An introduction to the tomographic picture of quantum mechanics," *Phys. Scr.* **79**, 065013 (2009).
7. O. V. Man'ko, and V. I. Man'ko, "Quantum states in probability representation and tomography," *J. Russ. Laser Res.*, **18**, 407–444 (1997).
8. V. I. Man'ko and R. Vilela Mendes, "Lyapunov exponent in quantum mechanics. A phase-space approach," *Physica D*, **145**, 330–348 (2000).
9. C. E. Shannon, "A mathematical theory of communication," *Bell. Tech. J.*, **27**, 379–423 (1948).
10. A. Rényi, *Probability Theory*, North-Holland, Amsterdam, 1970.
11. M. A. Man'ko, "Entropy of an analytic signal," *J. Russ. Laser Res.*, **27**, 405–413 (2006).
12. M. A. Man'ko and V. I. Man'ko, "Probability description and entropy of classical and quantum systems," *Found. Phys.*, **41**, 330–344 (2011) [DOI 10.1007/s10701-009-9403-9].
13. I. I. Hirschman, "A note on entropy," *Amer. J. Math.*, **79**, 152–156 (1957).
14. V. V. Dodonov, and V. I. Man'ko, *Invariants and the Evolution of Nonstationary Quantum Systems, Proceedings of the P. N. Lebedev Physical Institute*, Nauka, Moscow, 1987, Vol. 183 [translated by Nova Science, New York, 1989].
15. I. Białynicki-Birula, "Formulation of the uncertainty relations in terms of the Rényi entropies," *Phys. Rev. A*, **74**, 052101 (2006) [ArXiv quant-ph/0608116 v1].
16. M. A. Man'ko, V. I. Man'ko, S. De Nicola, and R. Fedele, "Probability representation and new entropic uncertainty relations for symplectic and optical tomograms" *Acta Phys. Hung. B*, **26**, 71–77 (2006).
17. M. Bellini, A. S. Coelho, S. N. Filippov, V. I. Man'ko, and A. Zavatta, "Towards higher precision and operational use of optical homodyne tomograms," *Phys. Rev. A*, **85**, 052129 (2012).
18. V. I. Man'ko and O. V. Man'ko, "Spin state tomography," *Zh. Éksp. Teor. Fiz.*, **112**, 796–800 (1997) [*J. Exp. Theor. Phys.*, **85**, 430–434 (1997)].
19. V. V. Dodonov and V. I. Man'ko, "Positive distribution description for spin states," *Phys. Lett. A*, **239**, 335–339 (1997).
20. J. Bertrand and P. Bertrand, "A tomographic approach to Wigner's function," *Found. Phys.*, **17**, 397–405 (1987).
21. K. Vogel and H. Risken, "Determination of quasiprobability distributions in terms of probability distributions for the rotated quadrature phase," *Phys. Rev. A*, **40**, 2847–2849 (1989).
22. V. I. Man'ko, G. Marmo, E. C. G. Sudarshan, and F. Zaccaria, "Entanglement structure of adjoint representation of unitary group and tomography of quantum states," *J. Russ. Laser Res.*, **24**, 507–543 (2003).
23. P. Albin, E. De Vito, and A. Toigo, "Quantum homodyne tomography as an informationally complete positive-operator-valued measure," *J. Phys. A: Math. Theor.*, **42**, 295302 (2009).
24. D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, "Measurement of the Wigner distribution and the density matrix of a light mode using optical homodyne tomography: Application to squeezed states and the vacuum," *Phys. Rev. Lett.*, **70**, 1244–1247 (1993).
25. V. I. Man'ko, G. Marmo, E. C. G. Sudarshan, and F. Zaccaria, "Positive maps of density matrix and a tomographic criterion of entanglement," *Phys. Lett. A*, **327**, 353–364 (2004).
26. O. Castaños, R. López-Peña, M. A. Man'ko, and V. I. Man'ko, "Kernel of star-product for spin tomograms," *J. Phys. A: Math. Gen.*, **36**, 4677–4688 (2003).
27. S. Mancini, P. Tombesi, and V. I. Man'ko, "Density matrix from photon-number tomography," *Europhys. Lett.*, **37**, 79–83 (1997).
28. S. A. Arkhipov and V. I. Man'ko, "Quantum transitions in the center-of-mass tomographic probability representation," *Phys. Rev. A* **71**, 012101 (2005).
29. V. I. Man'ko, G. Marmo, and C. Stornaiolo, "Tomographic entropy and cosmology," *Gen. Rel. Grav.*, **40**, 1449–1465 (2008).